Lifting Galois representations over arbitrary number fields

Yoshiyuki Tomiyama *

1 Introduction

Let $k$ be a finite field of characteristic $p \geq 3$. Let $K$ be a number field of finite degree over $\mathbb{Q}$ and $G_K$ its absolute Galois group $\text{Gal}(\overline{K}/K)$. We consider continuous representations

$$\bar{\rho} : G_K \to \text{GL}_2(k).$$

We investigate the existence of a lift of $\bar{\rho}$ to $W(k)$, the ring of Witt vectors of $k$. This problem has been motivated by a conjecture of Serre ([S]), that is, all odd absolutely irreducible continuous representations $\bar{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(k)$ are modular of prescribed weight, level and character. This predicts the existence of a lift to characteristic zero. This conjecture was proved by Khare and Wintenberger in [KW1,KW2]. In [K], Khare proved the existence of lifts to $W(k)$ for any $\bar{\rho} : G_K \to \text{GL}_2(k)$ which are reducible. Ramakrishna proved under very general conditions on $\bar{\rho}$ that there exist lifts to $W(k)$ for $K = \mathbb{Q}$ in [R1,R2]. Gee’s results ([G]) imply that there exist lifts to $W(k)$ for $p \geq 5$ and $K$ satisfying $[K(\mu_p) : K] \geq 3$, where $\mu_p$ is the group of $p$-th roots of unity. Böckle and Khare have proved the general $n$-dimensional case for function field in [BK]. In [H], Hamblen proved the general $n$-dimensional case for $K = \mathbb{Q}$. In this paper, we extend Theorem 1 of [R1] to arbitrary number fields. In particular, we will omit the condition $[K(\mu_p) : K] \geq 3$. Hence we can take the field $K$ to be $\mathbb{Q}(\mu_p)^+$, the totally real subfield of $\mathbb{Q}(\mu_p)$.

For a place $v$ of $K$, let $K_v$ be the completion of $K$ at $v$, and let $G_v$ be its absolute Galois group $\text{Gal}(\overline{K_v}/K_v)$. For each place $v$ of $K$, we fix an embedding $\overline{K} \subset \overline{K_v}$. This gives a corresponding continuous homomorphism $G_v \to G_K$. Let $\text{Ad}^0 \bar{\rho}$ be the set of all trace zero two-by-two matrices over $k$ with Galois action through $\bar{\rho}$ by conjugation. Our main result is the following:

*Graduate School of Mathematics, Kyushu University

\text{e-mail : y-tomiyama@math.kyushu-u.ac.jp}
Theorem. Let $K$ be a number field, and let $\bar{\rho} : G_K \to \text{GL}_2(k)$ be a continuous representation with coefficients in a finite field $k$ of characteristic $p \geq 7$. Assume that $H^2(G_v, \text{Ad}^0 \bar{\rho}) = 0$ for each places $v | p$. Then $\bar{\rho}$ lifts to a continuous representation $\rho : G_K \to \text{GL}_2(W(k))$ which is unramified outside a finite set of places of $K$.

2 A criterion for lifting problems and killing the Selmer group

Let $S$ denote a finite set of places of $K$ containing the places above $p$, the infinite places and the places at which $\bar{\rho}$ is ramified, and let $K_s$ denote the maximal algebraic extension of $K$ unramified outside $S$. Thus $\bar{\rho}$ factors through $\text{Gal}(K_s/K)$. Put $G_{K,S} = \text{Gal}(K_s/K)$.

Let $\mathcal{A}$ be the category of complete noetherian local rings $(R,\mathfrak{m}_R)$ with residue field $k$ where the morphisms are homomorphisms that induce the identity map on the residue field.

Fix a continuous homomorphism $\delta : G_{K,S} \to W(k)^\times$, and for every $(R,\mathfrak{m}_R) \in \mathcal{A}$ let $\delta_R$ be the composition $\delta_R : G_{K,S} \to W(k)^\times \to R^\times$. Suppose $\bar{\rho} : G_{K,S} \to \text{GL}_2(k)$ has $\det \bar{\rho} = \delta_k$.

By a $\delta$-lift (resp. $\delta|_{G_v}$-lift) of $\bar{\rho}$ (resp. $\bar{\rho}|_{G_v}$) we mean a continuous representation $\rho : G_{K,S} \to \text{GL}_2(R)$ (resp. $\rho_v : G_v \to \text{GL}_2(R)$) for some $(R,\mathfrak{m}_R) \in \mathcal{A}$ such that $\rho (\text{mod } \mathfrak{m}_R) = \bar{\rho}$ (resp. $\rho_v (\text{mod } \mathfrak{m}_R) = \bar{\rho}|_{G_v}$) and $\det \rho = \delta_R$ (resp. $\det \rho_v = \delta_R|_{G_v}$).

Definition 1 (Taylor’s conditions). For a place $v$ of $K$, we say that a pair $(\mathcal{C}_v,L_v)$, where $\mathcal{C}_v$ is a collection of $\delta|_{G_v}$-lifts of $\bar{\rho}|_{G_v}$, and $L_v$ is a subspace of $H^1(G_v, \text{Ad}^0 \bar{\rho})$, is locally admissible if it satisfies the following conditions:

(P1) $(k,\bar{\rho}|_{G_v}) \in \mathcal{C}_v$.

(P2) The set of $\delta|_{G_v}$-lifts in $\mathcal{C}_v$ to a fixed ring $(R,\mathfrak{m}_R) \in \mathcal{A}$ is closed under conjugation by elements of $1 + M_2(\mathfrak{m}_R)$.

(P3) If $(R,\rho) \in \mathcal{C}_v$ and $f : R \to S$ is a morphism in $\mathcal{A}$ then $(S, f \circ \rho) \in \mathcal{C}_v$.

(P4) Suppose that $(R_1,\rho_1)$ and $(R_2,\rho_2) \in \mathcal{C}_v$, and $I_1$ (resp. $I_2$) is an ideal of $R_1$ (resp. $R_2$) and that $\phi : R_1/I_1 \sim \rightarrow R_2/I_2$ is an isomorphism such that $\phi (\rho_1 (\text{mod } I_1)) = \rho_2 (\text{mod } I_2)$. Let $R_3$ be the fiber product of $R_1$ and $R_2$ over $R_1/I_1 \sim \rightarrow R_2/I_2$. Then $(R_3,\rho_1 \oplus \rho_2) \in \mathcal{C}_v$.

(P5) If $((R,\mathfrak{m}_R),\rho)$ is a $\delta|_{G_v}$-lift of $\bar{\rho}|_{G_v}$ such that each $(R/\mathfrak{m}_R^n,\rho (\text{mod } \mathfrak{m}_R^n)) \in \mathcal{C}_v$ then $(R,\rho) \in \mathcal{C}_v$.
Remark 1. Note that we do regard $\mathcal{C}_v$ as a functor from $\mathcal{A}$ to the category of sets.

Let $S_{\ell}$ be the subset of $S$ consisting of finite places. Throughout this section, suppose that for each $v \in S_{\ell}$ a locally admissible pair $(\mathcal{C}_v, L_v)$ is given.

Let $\overline{\chi}_p : G_K \to k^*$ be the mod $p$ cyclotomic character. For the $k[G_K]$-module $\text{Ad}^0 \overline{\rho}$, by $\text{Ad}^0 \overline{\rho}(i)$ for $i \in \mathbb{Z}$ we denote the twist of $\text{Ad}^0 \overline{\rho}$ by the $i$-th tensor power of $\overline{\chi}_p$, and by $\text{Ad}^0 \overline{\rho}^* := \text{Hom}(\text{Ad}^0 \overline{\rho}, k)$ we denote its dual representation. The $G_K$-equivariant trace pairing $\text{Ad}^0 \overline{\rho} \times \text{Ad}^0 \overline{\rho} \to k : (A, B) \mapsto \text{Trace}(AB)$ is perfect. In particular, $\text{Ad}^0 \overline{\rho} \cong \text{Ad}^0 \overline{\rho}^*$ as representations. Thus $\text{Ad}^0 \overline{\rho}(1) \cong \text{Ad}^0 \overline{\rho}^*(1)$ as representations. By the Tate local duality this induces a perfect pairing

$$H^1(G_v, \text{Ad}^0 \overline{\rho}) \times H^1(G_v, \text{Ad}^0 \overline{\rho}(1)) \to H^2(G_v, k(1)) \cong k.$$ 

Definition 2. A $\delta$-lift of type $(\mathcal{C}_v)_{v \in S_{\ell}}$ is a $\delta$-lift such that $\rho|_{G_v} \in \mathcal{C}_v$ for all $v \in S_{\ell}$.

Definition 3. We define the Selmer group $H^1_{L_v}(G_{K,S}, \text{Ad}^0 \overline{\rho})$ to be the kernel of the map

$$H^1(G_{K,S}, \text{Ad}^0 \overline{\rho}) \to \bigoplus_{v \in S_{\ell}} H^1(G_v, \text{Ad}^0 \overline{\rho})/L_v$$

and the dual Selmer group $H^1_{L_v^\perp}(G_{K,S}, \text{Ad}^0 \overline{\rho}(1))$ to be the kernel of the map

$$H^1(G_{K,S}, \text{Ad}^0 \overline{\rho}(1)) \to \bigoplus_{v \in S_{\ell}} H^1(G_v, \text{Ad}^0 \overline{\rho}(1))/L_v^\perp$$

where $L_v^\perp \subset H^1(G_v, \text{Ad}^0 \overline{\rho}(1))$ is the annihilator of $L_v \subset H^1(G_v, \text{Ad}^0 \overline{\rho})$ under the above pairing.
A criterion of Ramakrishna [R2] and Taylor [Ta] for a lifting from a fixed residual Galois representation to a \( p \)-adic Galois representation is the following:

**Proposition.** Keep the above notation and assumptions. If

\[
H^1_{\{L_v\}}(G_{K,S}, \Ad^0 \bar{\rho}(1)) = 0,
\]

then there exists a \( \delta \)-lift of \( \bar{\rho} \) to \( W(k) \) of type \((C_v)_{v \in S_\delta}\).

**Lemma.** Suppose that one is given locally admissible pairs \((C_v, L_v)_{v \in S_\delta}\) such that

\[
\sum_{v \in S_\delta} \dim_k L_v \geq \sum_{v \in S} \dim_k H^0(G_v, \Ad^0 \bar{\rho}).
\]

Then we can find a finite set of places \( T \supset S \) and locally admissible pairs \((C_v, L_v)_{v \in T \setminus S}\) such that

\[
H^1_{\{L_v\}}(G_{K,T}, \Ad^0 \bar{\rho}(1)) = 0.
\]

We can give appropriate locally admissible pairs under the assumption of Theorem. By using this pairs, Proposition and Lemma, we can prove Theorem.

**References**


4


