Torsion points of abelian varieties with values in infinite extensions over a $p$-adic field

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1 Introduction

Let $A$ be an abelian variety over a field $K$. For an algebraic field extension $L$ over $K$, we are interested in the problem whether the torsion part $A(L)_{\text{tors}}$ of $L$-rational points $A(L)$ of $A$ is finite or not. Let us consider the following two cases:

1. (Global case) $K$ is a finite extension of $\mathbb{Q}$.
2. (Local case) $K$ is a finite extension of a $p$-adic field $\mathbb{Q}_p$.

If $L$ is a finite extension over $K$, it is well-known that the torsion part of the group $A(L)_{\text{tors}}$ is finite (cf. [Mat], Thm. 7) in both cases. On the other hand, in general, we do not know whether $A(L)_{\text{tors}}$ is finite or infinite if $L$ is an infinite algebraic extension over $K$.

In this talk we deduce some of results related with such a finiteness problem for $A(L)_{\text{tors}}$. Our results in this paper is given in Theorem 2.2, Theorem 2.3, Corollary 3.9 and Corollary 3.10 bellow.

2 Finiteness of torsion points in the local case

In this section, we consider the finiteness of $A(L)_{\text{tors}}$ under the local case, that is, $K$ is a finite extension of $\mathbb{Q}_p$. In this case there are not many results than the global case. Some of known results in the global case are given in the next section.

As a well-known fact, Imai proved the following, which is a straightforward generalization of Theorem 3.1.

**Theorem 2.1** (Imai [Im]). If $A$ has potential good reduction over $K$, then $A(K(\mu_\infty))_{\text{tors}}$ is finite.

By connecting this theorem, Theorem 3.1 and Theorem 3.2, it seems to be natural to hope that $A(K(\mu_\infty))_{\text{tors}}$ is finite. Unfortunately, by the following theorem, $A(K(\mu_\infty))_{\text{tors}}$ can be an infinite group for some abelian variety $A$, even if we assume that $A$ has good reduction.

**Theorem 2.2.** Let $A$ be an abelian variety over $K$ which has potential ordinary good reduction. Let $L$ be an algebraic extension of $K$ with residue field $k_L$. 

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(1) Assume that the residue field of \( L(\mu_p^\infty) \) is a potential prime-to-\( p \) extension of \( k \) (in the sense below). Then \( A(L)[p^\infty] \) is finite.

(2) If \( L \) contains \( K(A[p]) \) and \( K(\mu_p^\infty) \), then \( k_L \) is a potential prime-to-\( p \) extension of \( k \) if and only if \( A(L)[p^\infty] \) is finite.

(3) Assume that \( L(\mu_p^\infty) \) is a Galois extension of \( K \) whose residue field is finite. Then \( A(L)[p^\infty] \) tors is finite.

Here we say that the field extension \( F \) over \( E \) is a potential prime-to-\( p \) extension if \( F \) is a union of finite extensions of degree prime-to-\( p \) over some finite extension of \( E \). By using this theorem, we can obtain the global case of this theorem immediately, see the last statement of this paper.

Next we consider the finiteness of \( A(L)[p^\infty] \) tors for some \( p \)-adic Lie extension \( L \) over \( K \). Let \( B \) be an semi-abelian variety over \( K \). We denote by \( K_B,p = K(\beta[p^\infty]) \) the field generated by the coordinates of all \( p \)-power torsion points of \( B \). For example \( K_{\mathbb{G}_m,p} \) is the cyclotomic field \( K(\mu_p^\infty) \) and hence Imai’s theorem is a result on the torsion points of \( A(K_{\mathbb{G}_m,p}) \). We now suppose \( A = E_1 \) and \( B = E_2 \) are elliptic curves and consider \( E_1(K_{E_2,p})[p^\infty] \). We shall note that, for any prime \( \ell \neq p \), the group \( E_1(K_{E_2,p})[\ell^\infty] \) is always finite since \( K_{E_1,\ell} \) is a pro-\( \ell \)-extension over some finite extension of \( K \).

**Theorem 2.3.** The finiteness of \( E_1(K_{E_2,p})[p^\infty] \) is as follows:

<table>
<thead>
<tr>
<th>( E_1 )</th>
<th>( E_2 )</th>
<th># ( E_1(K_{E_2,p})[p^\infty] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ord</td>
<td>ord</td>
<td>( \infty ) ( \ast_1 )</td>
</tr>
<tr>
<td>ss</td>
<td>ss</td>
<td>finite</td>
</tr>
<tr>
<td>mult</td>
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</tr>
<tr>
<td>FCM</td>
<td>ord</td>
<td>CM</td>
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<tr>
<td>ss</td>
<td>non-CM</td>
<td>finite</td>
</tr>
<tr>
<td>non-FCM</td>
<td>ss</td>
<td>FCM</td>
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<tr>
<td>split mult</td>
<td>any</td>
<td>( \ast_3 ) finite</td>
</tr>
<tr>
<td>non-split mult</td>
<td>any</td>
<td>( \ast_3 ) finite</td>
</tr>
</tbody>
</table>

Here “ord”, “ss”, “mult”, “CM” and “FCM” in the above table stand for ordinary, supersingular, multiplicative, complex multiplication and formal complex multiplication, respectively. The symbols \( \ast_1, \ast_2 \) and \( \ast_3 \) in the table imply the followings:

\( \ast_1 \cdot \cdot \cdot \): ordinary good reduction
\( \Rightarrow E_1(K_{E_2,p})[p^\infty] \) is infinite in many cases.

\( \ast_2 \cdot \cdot \cdot \): supersingular good reduction with formal complex multiplication
\( E_2 \): ordinary good reduction with complex multiplication
\( \Rightarrow E_1(K_{E_2,p})[p^\infty] \) is finite in all cases.

\( \ast_3 \cdot \cdot \cdot \): supersingular good reduction without formal complex multiplication
\( \Rightarrow E_1(K_{E_2,p})[p^\infty] \) may be finite or infinite (case by case).
3 Finiteness of torsion points in the global case

In this section we line up some known facts about the finiteness of torsion points of abelian variety $A$ defined over an algebraic number field $K$. Fix an algebraic closure $\bar{K}$ of $K$. Let $K^{ab}$ be the maximal abelian extension of $K$ in $\bar{K}$. For each positive integer $m$, we write $K(\mu_m)$ for the subfield of $\bar{K}$ obtained by adjoining to $K$ all $m$-th roots of unity. We denote $\bigcup_m K(\mu_m)$ by $K(\mu_\infty)$, where $m$ runs through all positive integers. For example $\mathbb{Q}^{ab}$ is equal to $\mathbb{Q}(\mu_\infty)$ by the theorem of Kronecker-Weber. For each prime number $p$, we write $K(\mu_p^{\infty})$ by the subfield of $\bar{K}$ obtained by adjoining to $K$ all $p$-power roots of unity.

In the cyclotomic case, the following theorem is known:

**Theorem 3.1 (Imai [Im] and Serre [Se1]).** The group $A(K(\mu_p^{\infty}))^{\text{tors}}$ is finite for any prime number $p$.

By influenced the Mazur’s paper [Maz], Imai and Serre proved the above theorem independently. As a general result of Theorem 3.1, Ribet proved that the following:

**Theorem 3.2 (Ribet [Ri]).** The group $A(K(\mu_\infty))^{\text{tors}}$ is finite.

We shall remark that the statement of this theorem is not true for the local case, that is, $A'(K'(\mu_\infty))^{\text{tors}}$ is infinite in many cases for some abelian variety $A'$ over a $p$-adic field $K'$ even if we assume that $A'$ has good reduction over $K'$ (this can be checked easily from Theorem 2.2).

There is more information about the relation with cyclotomic fields and the finiteness of torsion points. Let $\text{End}_K(A)$ be the ring of all endomorphisms of $A$ defined over $K$. We write $\text{End}^0(A)$ for the finite-dimensional semisimple $\mathbb{Q}$-algebra $\text{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. By $\text{Lie}(A)$, we denote the tangent space to $A$ at the origin. Consider the following conditions:

(i) there is a discrete valuation $v$ on $K$ such that $A$ has potentially purely multiplicative reduction at $v$;

(ii) $K$ does not contain a CM-field;

(iii) the Hodge group of $A$ is semisimple;

(iv) the center $F$ of $\text{End}^0(A)$ is a CM-field and the pair $(A, F)$ is of Weil type, that is, the $F \otimes \mathbb{Q}$ $K$-module $\text{Lie}(A)$ is free.

**Theorem 3.3 (Zarhin [Za3], Thm. 0.3).** Assume that the abelian variety $A$ and the number field $K$ satisfy at least one of the conditions (i), (ii), (iii) and (iv) above. If the intersection of $L$ and $K(\mu_\infty)$ is a finite extension of $K$, then $A(L)^{\text{tors}}$ is finite.

Zarhin has also proved that

**Theorem 3.4 (Zarhin [Za1], Thm. 6).** Assume the center of $\text{End}^0(A)$ is a direct sum of totally real number fields. If $L$ contains only finitely many roots of unity, then $A(L)^{\text{tors}}$ is finite.

For the relation of the maximal abelian extension and the torsion part, next theorems are known:

**Theorem 3.5 (Bogomolov).** If the intersection of $L$ and $K^{ab}$ is a finite extension of $K$, then $A(L)^{\text{tors}}$ is finite.
This theorem is proved in Séminaire Delange-Pisot-Poitou, mai 1982, Paris. For this proof, see [Col].

We say that a simple abelian variety $B$ over $K$ of dimension $d$ is of CM-type over $K$ if $\text{End}^0(B)$ is a number field of dimension $2d$ over $\mathbb{Q}$. If $A$ is of CM-type, then the torsion subgroups of $A(K)$ and $A(K^{ab})$ coincide ([ST]). In particular, the torsion part of $A(K^{ab})$ is infinite if $A$ is of CM-type. In fact the converse is true;

**Theorem 3.6** (Zarhin [Za1], Thm. 1). *If an abelian variety $A$ over $K$ is simple, then $A(K^{ab})_{\text{tors}}$ is finite if and only if $A$ is not of CM-type over $K$. In general, $A(K^{ab})_{\text{tors}}$ is finite if and only if $A$ does not contain non-zero simple abelian subvarieties over $K$ of CM-type over $K$.*

Let $\ell$ be a prime number. The following theorems are related to $\ell$-adic Lie extensions (e.g. the extension $K(A(\ell^\infty))$ over $K$).

**Theorem 3.7** (Zarhin [Za2], Thm. 0.10). *Assume that $L$ is an infinite Galois extension whose Galois group $\text{Gal}(L/K)$ is isomorphic to a compact $\ell$-adic Lie group. Then
(1) the group $A(L)[p^\infty]$ is finite for any prime number $p$ different from $\ell$.
(2) If $A(L)[p^\infty]$ does not vanish for infinitely many primes $p$, then $A$ is of CM-type over $L$.***

**Theorem 3.8** (Greenberg [Gr], Prop. 1.2). *Let $L$ be a Galois extension of $K$ whose Galois group $\text{Gal}(L/K)$ is isomorphic to a $\ell$-adic Lie group. Then $A(L)[p^\infty]$ is finite if one of the following conditions is satisfied.
(1) There exists a nonarchimedian prime $\eta$ of $K$ not lying over $p$ such that the corresponding residue field $k_\eta$ is finite.
(2) The Lie algebra $\text{Lie}(\text{Gal}(L/K))$ is solvable, $A$ has potential ordinary good reduction at all primes of $L$ lying above $p$, and the residue field $k_\eta$ at $\eta$ is finite for all primes $\eta$ of $K$ lying above $p$.
(3) The Lie algebra $\text{Lie}(\text{Gal}(L/K))$ is semisimple, that is, it is a direct product of simple, non-abelian Lie algebras.*

Let $v$ be a finite place of $K$. For any finite extension $K'$ of $K$ and any finite place $v'$ of $K'$ above $v$, we denote the completion of $K'$ at $v'$ by $K'_v$. More generally, for any algebraic extension $L$ and any place $w$ above $v$, we denote

$$L_w := \bigcup_{K'} K'_v,$$

where $K'$ runs through all the finite extensions of $K$ in $L$ and $v'$ is the unique place of $K'$ under $w$. Note that the residue field $k_{L_w}$ of $L_w$ is $\bigcup_{K'} k_{K'_v}$.

As corollaries of Thm. 2.2, we can see the “global cases” below immediately.

**Corollary 3.9.** *Let $K, L, A$ be as above. Assume that there exist places $v$ of $K$ above $p$ and $w_\infty$ of $L(\mu_p^\infty)$ above $v$ satisfying the following properties:
(i) The residue field $k_{w_\infty}$ of $L(\mu_p^\infty)$ at $w_\infty$ is a potential prime-to-$p$ extension of the residue field $k_v$ of $K$ at $v$.
(ii) $A$ has potential ordinary good reduction at $v$.
Then $A(L)[p^\infty]$ is finite.*
Corollary 3.10. Let $K, L, A$ be as above. Assume that $L(\mu_p^{\infty})$ is a Galois extension of $K$, and there exist places $v$ of $K$ above $p$ and $w_{\infty}$ of $L(\mu_p^{\infty})$ above $v$ satisfying the following properties:

(i) The residue field $k_{w_{\infty}}$ of $L(\mu_p^{\infty})$ at $w_{\infty}$ is finite.
(ii) $A$ has potential ordinary good reduction at $v$.

Then $A(L)_{\text{tors}}$ is finite.

If we always assume that $L$ contains all $p$-power roots of unity, these corollaries are generalizations of the result of Greenberg given in the above.

References


