Some remarks on the pseudo-nullity conjecture for zero Selmer groups of elliptic curves

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1 zero Selmer group and the conjecture

Let $E$ be an elliptic curve over $\mathbb{Q}$, and $p$ an odd prime. Let $F$ be a finite extension of $\mathbb{Q}$ and $\Sigma$ some finite set of primes of $F$. Given a local condition $L = \{L_v\}$, we have (p-part of) the Selmer group $\text{Sel}_L(E/F)$ associated to the local condition such that the following sequence is exact:

$$0 \to \text{Sel}_L(E/F) \to H^1(F_{\Sigma}/F, E_{p^\infty}) \to \bigoplus_{v \in \Sigma} H^1(F_v, E_{p^\infty})/L_v$$

In the case $L_v = E(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ for $v | p$, the Selmer group is the classical one. We are going to consider the Selmer group with the local condition $L_v = \{0\}$, which is denoted by $\text{Sel}_0(E/F)$. This Selmer group is called zero Selmer group. The zero Selmer group appears in Kato-Kurihara’s formulation of the main conjecture without $p$-adic $L$ function in cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ ([9] Conjecture 6.1).

Let $K_\infty = \bigcup K_n$ be a Galois extension of a number field $K$ with Galois group $G = G(K_\infty/K)$, and $S$ a finite set of primes of $K$ satisfying some condition. Let $\text{Sel}_0(E/K_n)$ denote the kernel of the restriction map

$$H^1(K_S/K_n, E_{p^\infty}) \to \bigoplus_{v \in S} H^1(K_{n,v}, E_{p^\infty})$$

for $K \subset K_n \subset K_\infty$ such that $[K_n : K] < \infty$ and $\text{Sel}_0(E/K_\infty) := \lim \text{Sel}_0(E/K_n)$. The fine Selmer group $\text{Sel}_0(E/K_\infty)$ is a subgroup of the classical Selmer group $\text{Sel}(E/K_\infty)$. Coates and Sujatha made the following conjectures:

Conjecture 1.1 (Conjecture A in [2]) $\text{Sel}_0(E/K(\mu_{p^\infty}))^\vee$ is a finitely generated $\mathbb{Z}_p$-module.

Conjecture 1.2 (Conjecture B in [2]) For any admissible pro-p $p$-adic Lie extension $K_\infty/K$ of dimension $\geq 2$, $\text{Sel}_0(E/K_\infty)^\vee$ is a pseudo-null $\Lambda(G)$-module.

We focus on the second conjecture, which we are going to call the "pseudo-nullity conjecture" in this article. However, this conjecture which was proposed in 2005 has had no nontrivial example. We will point out that there are some rather trivial examples of elliptic curves and prime numbers which satisfy the conjecture.

Definition 1.3 • For a $\mathbb{Z}_p$-module $M$, the module $M^\vee$ denotes the Pontryagin dual of $M$, i.e., $M^\vee = \text{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$.  

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• For a compact \( p \)-adic Lie group \( G \), the so-called Iwasawa algebra \( \Lambda(G) \) is defined to be \( \lim_{\leftarrow H \subset G} \mathbb{Z}_p[G/H] \), where \( H \) runs over normal open subgroups of \( G \).

\[ E^i(M) := \text{Ext}^i_{\Lambda(G)}(M, \Lambda(G)). \]

• Let \( K \) be a number field. A Galois extension \( K/K_\infty \) is called an admissible \( p \)-adic Lie extension with Galois group \( G = G(K_\infty/K) \), if \( G \) is a compact \( p \)-adic Lie group which has no element of order \( p \), and \( K_\infty \) contains the cyclotomic \( \mathbb{Z}_p \)-extension of \( K \), and only a finite number of primes of \( K \) ramify in \( K_\infty \).

A word on “pseudo-null” thing: In the case \( \Lambda(G) \) is a commutative ring, a torsion \( \Lambda(G) \)-module is called pseudo-null if it is annihilated by an ideal of height at least 2 in \( \Lambda(G) \). In the case \( G \simeq \mathbb{Z}_p \), a finitely generated torsion \( \Lambda(G) \) is pseudo-null iff it is finite. This property turns out equivalent to the vanishing of Ext-groups: \( \text{Ext}^i_{\Lambda(G)}(M, \Lambda(G)) = 0 \) for \( i = 0, 1 \). The latter property can be applied to the cases when \( \Lambda(G) \) is not commutative. So, we take the following definition of pseudo-null:

**Definition 1.4** A finitely generated \( \Lambda(G) \)-module \( M \) is said to be pseudo-null if \( E^i(M) = 0 \) for \( i = 0, 1 \).

2 Some partial results

In the sequel, we consider the extension \( K_\infty/K \) such that \( K_\infty = \mathbb{Q}(E_p^\infty), K = \mathbb{Q}(\mu_p) \).

Let \( H = \text{Gal}(K_\infty/K(\mu_{p^n})) \). There is the fact that a finitely generated \( \Lambda(G) \)-module \( M \) is pseudo-null iff \( M \) is a finitely generated \( \Lambda(H) \)-torsion module, i.e., iff \( \text{rank}_{\Lambda(H)}(M) = 0 \).

**Example 2.1** ([2]) Let \( E \) be an elliptic curve over \( \mathbb{Q} \) defined by the equation

\[ y^2 + xy = x^3 - x - 1 \]

which is the curve of conductor 294 in Cremona’s table [4]. Then it was shown in [2] that \( \text{rank}_{\Lambda(H)}(\text{Sel}_0(E/K_\infty)^\vee) = 0 \) or 2.

**Remark 2.2** There is an isomorphism:

\[ \text{Sel}_0(E/K_\infty) = \text{Hom}(\text{Gal}(L'_\infty/K_\infty), E_p^\infty) \]

\[ \cong \text{Hom}(\text{Gal}(L'_\infty/K_\infty), \mathbb{Q}_p/\mathbb{Z}_p) \oplus \text{Hom}(\text{Gal}(L'_\infty/K_\infty), \mathbb{Q}_p/\mathbb{Z}_p) \]

where \( L'_\infty \) is the maximal \( p \)-extension of \( K_\infty \) in which the prime over \( S \) splits completely. Therefore \( \text{rank}_{\Lambda(H)}(\text{Sel}_0(E/K_\infty)^\vee) \) is always even.

**Example 2.3** ([2]) Let \( E \) be an elliptic curve over \( \mathbb{Q} \) defined by the equation

\[ y^2 + xy = x^3 - 3x - 3 \]

which is the curve of conductor 150 in Cremona’s table [4], and does not admit complex multiplication. Then it was shown in [2] that if \( E(K_\infty) \) has a point of infinite order, which is likely, then \( \text{rank}_{\Lambda(H)}(\text{Sel}_0(E/K_\infty)^\vee) = 0 \), i.e., the pseudo-nullity conjecture holds.
3 Examples of trivial zero Selmer groups

From the Poitou-Tate duality, we have the following injective map (see for example [2] (43))

\[ \text{Sel}_0(E/K_\infty)^\vee \hookrightarrow H^2_{Iw}(T) \]  

where \( H^2_{Iw}(T) = \lim_{\rightarrow \operatorname{cor}} H^2(K_S/K_n, T_p(E)) \). Therefore it is sufficient to show that \( H^2_{Iw}(T) \) is pseudo-null.

Further we can prove

**Proposition 3.1** ([11]) Let \( K_\infty/K \) be any admissible pro-p p-adic Lie extension of dimension \( \geq 2 \). Assume \( H^2(K_S/K_\infty, E_{p^\infty}) = 0 \). Then the pseudonullity conjecture is true \( \iff H^1(K_S/K_n, E_{p^\infty})^\vee \) has no nontrivial torsion submodule.

Then there are some examples available which indeed satisfy the pseudonullity conjecture. The following is the only example with non-CM elliptic curve:

**Example 3.2** ([11]) Let \( E \) be the elliptic curve defined by

\[ y^2 + xy = x^3 + x^2 - 2x - 7 \]  

which is 121C1 in Cremona’s table [4] and does not admit complex multiplication, and take \( p = 11 \).

Then \( \text{Sel}_0(E/Q(E_{11^\infty}))^\vee \) is a pseudo-null \( \Lambda(G) \)-module.

There are some more examples with CM elliptic curves. One of which is:

**Example 3.3** Let \( E \) be the elliptic curve defined by

\[ y^2 + y = x^3 + 20 \]  

which is 243A2 in [4], and take \( p = 3 \). Then \( \text{Sel}_0(E/Q(E_{3\infty}))^\vee \) is a pseudo-null \( \Lambda(G) \)-module.

Actually these zero Selmer groups in the examples turn out trivial.

4 Final Remarks

- Hachimori and Sharifi have shown ([6]) that there are some admissible pro-p Lie extension of CM fields \( F_\infty/F \) such that \( X_{F_\infty} = \varprojlim A_{F_\infty}(p) \) (projective limit is taken with respect to norm maps) is not pseudo-null, where \( A_{F_\infty}(p) \) is the p-sylow subgroup of the ideal class group of \( F_\infty \). They suggest that we should restrict to what can be called admissible p-adic Lie extensions "coming from algebraic geometry" for \( X_{F_\infty} \) to be pseudo-null.

In the sequel, let \( K_\infty = Q(E_{p^\infty}) \) and assume \( E \) has complex multiplication.

- Because of the isomorphism

\[ \text{Sel}_0(E/K_\infty) = \text{Hom}(\text{Gal}(L_\infty'/K_\infty), E_{p^\infty}) \]

and \( \text{Gal}(L_\infty'/K_\infty) \) is a quotient of \( \text{Gal}(L_\infty/K_\infty) \) where \( L_\infty \) is the maximal abelian pro-p-extension of \( K_\infty \), the pseudo-nullity conjecture of Coates and Sujatha has a lot to do with the following conjecture due to Greenberg:
Conjecture 4.1 ([5] Conjecture (3.5)) Let $\tilde{K}$ denote the compositum of all $\mathbb{Z}_p$-extension of $K$, and let $\tilde{L}$ denote the maximal abelian unramified pro-$p$ extension of $\tilde{K}$, then $\tilde{X} = \text{Gal}(\tilde{L}/\tilde{K})$, as a $\Lambda(\text{Gal}(\tilde{K}/K))$-module, is pseudo-null.

- Coates and Sujatha observed in [3] that the pseudo-nullity conjecture for CM elliptic curves would simplify proof of the two-variable main conjecture.

- Minhyong Kim ([8]) made observations that for a CM elliptic curve $E$, one of his conjecture (Conjecture 1 in [8]) on Selmer varieties for the elliptic curve minus origin follows from the pseudo-nullity conjecture for $E$.

References