TRACES OF CLASS INVARIANTS

DAEYEOL JEON, SOON-YI KANG*, CHANG HEON KIM

Abstract. Zagier showed that the traces of singular values of the modular j-invariant are Fourier coefficients of a weakly holomorphic modular form of weight 3/2 and Bruinier and Funke generalized this result by defining modular traces of arbitrary weakly holomorphic modular functions so that they are Fourier coefficients of the holomorphic part of a harmonic weak Maass form. In this work, we present Galois traces of class invariants which are singular values of the modular functions that generate Hilbert class fields of imaginary quadratic fields and show that they coincide with modular traces of the singular values. As a result, we can show that there is a divisibility property on the Galois traces of the class invariants.

1. Introduction

The classical j-invariant is defined for $z$ in the complex upper half plane $\mathbb{H}$ by

$$j(z) := \frac{(1 + 240 \sum_{n=1}^{\infty} \sum_{m|n} m^3 q^n)^3}{q \prod_{n=1}^{\infty} (1 - q^n)^{24}} = q^{-1} + 744 + 196888q + 21493760q^2 + \cdots,$$

where $q = e^{2\pi i z}$. If $K = \mathbb{Q}(\theta)$ is an imaginary quadratic number field with the discriminant $-D$, then for the ring of integers $O = \mathbb{Z}[\theta]$, $j(O) = j(\theta)$ generates the Hilbert class field $H_K$ over $K$ by the classical theory of complex multiplication. The theory also says that the conjugates of $j(\theta)$ under the action of $\text{Gal}(H_K/K)$ are singular moduli $j(\tau)$, where

$$\theta := \begin{cases} \frac{i\sqrt{D}}{2}, & \text{if } D \equiv 0 \pmod{4}; \\ \frac{-1+i\sqrt{D}}{2}, & \text{if } D \equiv 3 \pmod{4}, \end{cases}$$

then for the ring of integers $O = \mathbb{Z}[\theta]$, $j(O) = j(\theta)$ generates the Hilbert class field $H_K$ over $K$ by the classical theory of complex multiplication. The theory also says that the conjugates of $j(\theta)$ under the action of $\text{Gal}(H_K/K)$ are singular moduli $j(\tau)$, where

$$\tau := \tau_Q := \frac{-b + i\sqrt{D}}{2a}$$

is a CM point associated to a positive definite integral binary quadratic form

$$Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$$

with discriminant $-D = b^2 - 4ac$. The trace of $j(\theta)$, that is the sum of the conjugates of $j(\theta)$, is an ordinary integer and is an interesting arithmetic function of $D$.

As the singular modulus $j(\tau_Q)$ depends only on the equivalence class of $Q$ under the action of $\Gamma(1) := \text{PSL}_2(\mathbb{Z})$, we may define the trace of $j(\theta)$ as

$$t_j(D) := \sum_{Q \in Q_D/\Gamma(1)} \frac{j(\tau_Q) - 744}{|\Gamma(1)Q|},$$

where $Q_D$ denotes the set of positive definite integral binary quadratic forms with discriminant $-D$ and $\Gamma(1)_Q$ denotes the isotropy subgroup of $Q$ in $\Gamma(1)$. After Zagier [4] proved that $t_j(D)$'s...
are Fourier coefficients of a weakly holomorphic modular form of weight 3/2 on $\Gamma_0(4)$, the trace of singular moduli has been the subject of number of works. In particular, Bruinier and Funke [1] showed that the generating function of modular traces of any weakly holomorphic modular function is the holomorphic part of a harmonic weak Maass form of weight 3/2 and if the constant terms in the Fourier expansions of the weakly holomorphic function vanish at all cusps, then the generating function of its modular traces is a weakly holomorphic modular form of weight 3/2.

Following Weber, for a modular function $f$, if $K(f(\theta)) = K(j(\theta))$, we call $f(\theta)$ a class invariant. As the Galois group $Gal(K(f(\theta))/K)$ acts on $f(\theta)$, we can find a Galois trace of the value. We will so discuss how the Galois trace and modular trace of class invariants are related with each other.

2. Weber Functions

Recall that the normalized Eisenstein series

$$g_2(z) = 60 \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(m + nz)^4}$$

$$g_3(z) = 140 \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(m + nz)^6}$$

are modular forms of weights 4 and 6, respectively. The Dedekind-eta function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \text{with} \quad q = e^{2\pi i z}$$

is holomorphic and non-zero for $z$ in the complex upper half plane $\mathbb{H}$ and $\Delta(z) = \eta^{24}(z)$ is modular form of weight 12 with no poles or zeros on $\mathbb{H}$. The classical $j$-invariant is defined for $z \in \mathbb{H}$ by

$$j(z) = 12^3 \frac{g_2^3(z)}{(2\pi)^{12} \Delta(z)} = 12^3 + 6^6 \frac{g_3^2(z)}{(2\pi)^{12} \Delta(z)}$$

is invariant under the group $\Gamma(1) = \text{PSL}_2(\mathbb{Z})$. All the modular groups discussed in this paper are subgroups of $\Gamma(1)$. We define Weber functions

$$(2.1) \quad \gamma_2(z) = \frac{12g_2(z)}{(2\pi)^4 \eta^8(z)}, \quad \gamma_3(z) = \frac{216g_3(z)}{(2\pi)^6 \eta^{12}(z)}$$

$$(2.2) \quad f(z) = \zeta_{48}^{-1} \frac{\eta(z)}{\eta(z)}, \quad \hat{f}_2(z) = \frac{\eta(\sqrt{z})}{\eta(z)}, \quad \hat{f}_3(z) = \sqrt{2} \frac{\eta(2z)}{\eta(z)}.$$ 

For the generating matrices $S, T \in \Gamma(1)$ given by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the transformation rules $\eta \circ S(z) = \sqrt{-iz} \eta(z)$ and $\eta \circ T(z) = \zeta_{24\gamma} \eta(z)$ hold. Hence

$$(2.3) \quad (\gamma_2, \gamma_3) \circ S = (\gamma_2, -\gamma_3) \quad (\gamma_2, \gamma_3) \circ T = (\zeta_{48}^{-1} \gamma_2, -\gamma_3),$$

$$(2.4) \quad (f, \hat{f}_2, \hat{f}_3) \circ S = (f, \hat{f}_2, \hat{f}_3) \quad (f, \hat{f}_2, \hat{f}_3) \circ T = (\zeta_{48}^{-1} f, \zeta_{48}^{-1} \hat{f}_2, \zeta_{48}^{-1} \hat{f}_3).$$
Therefore, $\gamma_2$ is $\Gamma(3)$-invariant, $\gamma_3$ is $\Gamma(2)$-invariant, and all three $f$ functions are $\Gamma(48)$-invariant.

Moreover, these Weber functions have the following properties.

**Lemma 2.1.** The constant coefficients of all Weber functions at all cusps vanish.

**Lemma 2.2.** $f_1$, $f_2^2$, and $f_3^2$ are $\Gamma_0(48)$-invariant.

3. **Galois action**

We will follow Gee [2] where she utilized Shimura reciprocity law to find Galois actions of class invariants. Let $K$ be an imaginary quadratic field of discriminant $-D$ with the ring of integers $\mathcal{O} = \mathbb{Z}[\theta]$. For an ideal $a \subset \mathcal{O}$, the formula

$$a : j(\mathcal{O}) \mapsto j(a^{-1})$$

gives the action of the Artin symbol for $a$ on the class group $\text{Cl}(\mathcal{O})$. Every primitive reduced quadratic form of discriminant $-D$ corresponds uniquely with an ideal class in $\text{Cl}(\mathcal{O})$. If $[a, b, c]$ is a primitive form of discriminant $-D$, then for $\tau = \frac{-b + \sqrt{-D}}{2a}$ the $\mathbb{Z}$-lattice $L = [a, a\tau]$ is an integral $\mathcal{O}$-ideal. The Galois action of the Artin symbol for $[a, -b, c]$ on $K(j(\theta))/K$ is given by

$$j(\theta)^{[a, -b, c]} = j(\tau).$$

**Theorem 3.1.** [2, Lemma 20] Let $\mathbb{Z}[\theta]$ be the ring of integers of an imaginary quadratic number field $K$ of discriminant $-D$ and let $[a, b, c]$ be a primitive quadratic form of discriminant $-D$. If $\theta = \frac{-B + \sqrt{-D}}{2a}$ as defined in (1.1) and $\tau = \frac{-b + \sqrt{-D}}{2a}$. Let $u_\tau = (u_p)_p$ be defined according to the local formulas for $u_p \in \text{GL}_2(\mathbb{Z}_p)$ given as follows: For $-D \equiv 0 \pmod{4}$, $u_p$ is given by

$$u_p = \begin{cases} 
\left( \begin{array}{cc}
\frac{a}{p} & \frac{b}{p} \\
0 & 1
\end{array} \right), & \text{if } p \nmid a; \\
\left( \begin{array}{cc}
\frac{-b}{p} & -c \\
1 & 0
\end{array} \right), & \text{if } p \mid a \text{ and } p \nmid c; \\
\left( \begin{array}{cc}
\frac{-b}{p} & -c \\
1 & -1
\end{array} \right), & \text{if } p \mid a \text{ and } p \mid c,
\end{cases}$$

and for $-D \equiv 1 \pmod{4}$, $u_p$ is given by

$$u_p = \begin{cases} 
\left( \begin{array}{cc}
\frac{a}{p} & \frac{b-1}{p} \\
0 & 1
\end{array} \right), & \text{if } p \nmid a; \\
\left( \begin{array}{cc}
\frac{-b-1}{2} & -c \\
1 & 0
\end{array} \right), & \text{if } p \mid a \text{ and } p \nmid c; \\
\left( \begin{array}{cc}
\frac{-b-1}{2} & -a \\
1 & \frac{b-1}{2}
\end{array} \right), & \text{if } p \mid a \text{ and } p \mid c.
\end{cases}$$

It follows that

$$h(\theta)^{[a, -b, c]} = h^{u_\tau}(\tau)$$

for any automorphic function $h$ such that $h(\theta) \in K(j(\theta))$.

For $-D$ which is congruent to a square modulo $4N$, we define

$$\mathcal{Q}_{D,N} = \{[a, b, c] \in \mathcal{Q}_D \mid a \equiv 0 \pmod{N}\}$$

$$\mathcal{Q}_{D,N,\beta} = \{[a, b, c] \in \mathcal{Q}_{D,N} \mid b \equiv \beta \pmod{2N}\}$$
on which \( \Gamma_0(N) \) acts. Similarly, for \(-D\) that is congruent to a square modulo \(4N^2\), we define

\[
\mathcal{Q}_{D,(N)} = \{ [a, b, c] \in \mathcal{Q}_{D,N} \mid a \equiv c \equiv 0 \pmod{N} \}
\]

\[
\mathcal{Q}_{D,(N),\beta} = \{ [a, b, c] \in \mathcal{Q}_{D,(N)} \mid b \equiv \beta \pmod{2N^2} \}
\]
on which \( \Gamma_0^0(N) \) acts. It is known from [3] that there is a canonical bijection between \( \mathcal{Q}_{D,N,\beta}/\Gamma_0^0(N) \) and \( \mathcal{Q}_D/\Gamma(1) \). We also note that there is a similar relation between \( \mathcal{Q}_{D,(N),\beta}/\Gamma_0^0(N) \) and \( \mathcal{Q}_{D,N,\beta}/\Gamma_0(N) \):

**Lemma 3.2.** Assume that \( D \) is not divisible as a discriminant by the square of any prime dividing \( N \).

(i) There is a canonical bijection between \( \mathcal{Q}_{D,(N)}/\Gamma_0^0(N) \) and \( \mathcal{Q}_{D,N}/\Gamma_0^0(N) \).

(ii) There is a canonical bijection between \( \mathcal{Q}_{D,(N),\beta}/\Gamma_0^0(N) \) and \( \mathcal{Q}_{D,N,\beta}/\Gamma_0(N) \).

**Lemma 3.3.** Suppose \(-D\) is an imaginary quadratic discriminant such that \(3 \nmid D\). We let \( \theta = \frac{-B + \sqrt{-D}}{2} \) as defined in (1.1) and let \( Q = [a, b, c] \) be a primitive quadratic form of discriminant \(-D\). The action of the form class group on \( \zeta_3^B \gamma_2(\theta) \) is given by the formula

\[
(\zeta_3^B \gamma_2)[a,-b,c](\theta) = \left\{ \begin{array}{ll}
\zeta_3^{ab} \gamma_2(\tau), & \text{if } 3 \nmid a; \\
\zeta_3^{bc} \gamma_2(\tau), & \text{if } 3 \mid a \text{ and } 3 \nmid c; \\
\gamma_2(\tau), & \text{if } 3 \mid a \text{ and } 3 \mid c.
\end{array} \right.
\]

**Proposition 3.4.** [2, Proposition 22] Suppose \(-D \equiv 1 \pmod{8}\) is an imaginary quadratic discriminant such that \(-D \not\equiv 0 \pmod{3}\). We let \( \theta = \frac{-1 + \sqrt{-D}}{2} \) and let \( Q = [a, b, c] \) be a primitive quadratic form of discriminant \(-D\). The action of the form class group on \( \zeta_48 \gamma_2(\theta) \) is given by the formula

\[
\zeta_48^+[a,-b,c](\theta) = \left\{ \begin{array}{ll}
\zeta_48^{b(a-c+a^2c)} \gamma_2(\tau_Q), & \text{if } 2 \nmid a; \\
\zeta_48^{b(a-c-a^2c)} \gamma_2(\tau_Q), & \text{if } 2 \mid a \text{ and } 2 \nmid c; \\
(-1)^{a-1} \zeta_48^{b(a-c-a^2c)} \gamma_2(\tau_Q), & \text{if } 2 \mid a \text{ and } 2 \mid c.
\end{array} \right.
\]

**Lemma 3.5.** Let \(-D\) be a fundamental discriminant such that \(-D\) is congruent to a square modulo 36 but not divisible by 3. If \( \theta = \frac{-B + \sqrt{-D}}{2} \) is defined as in (1.1) again and \( \tau_Q \) is a CM point given in (1.2), then the Galois trace of \( \zeta_3^B \gamma_2(\theta) \) is equal to

\[
\text{Gal}^T \tau_{\zeta_3^B \gamma_2(\theta)} := \sum_{Q \in \mathcal{Q}_{D,(3),1}/\Gamma_0^0(3)} \gamma_2(\tau_Q).
\]

**Lemma 3.6.** Let \(-D\) be a fundamental discriminant such that \(-D\) is congruent to a square modulo 192 but not divisible by 2 or 3. If \( \theta = \frac{-1 + \sqrt{-D}}{2} \) and \( \tau_Q \) is a CM point given in (1.2), then the Galois trace of \( \zeta_48 \gamma_2(\theta) \) is equal to

\[
\text{Gal}^T \tau_{\zeta_48 \gamma_2(\theta)} := \sum_{Q \in \mathcal{Q}_{D,(48)}/\Gamma_0^0(48)} \gamma(\tau_Q).
\]

4. Modular Trace and Galois Trace of Class Invariants

**Lemma 4.1.** Assume that \( D \) is a discriminant.

(i) If \( 2 \mid D \), then \( t_{\gamma_2}(D) = 0 \).

(ii) If \( 3 \mid D \), then \( t_{\gamma_2}(D) = 0 \).
Lemma 4.2. If $2, 3|D$, then $t_1(D) = 0$.

Therefore, it follows from the lemmas above and results in the previous sections, we have

**Theorem 4.3.** The generating function of the Galois traces of $\zeta_3 B_2(\theta)$ is a weakly holomorphic modular form of weight $3/2$ on $\Gamma(36)$.

**Theorem 4.4.** The generating function of the Galois traces of $\zeta_{48} f_2(\theta)$ is a weakly holomorphic modular form of weight $3/2$ on $\Gamma_0^0(9216)$.

**REFERENCES**


Kongju National University, Kongju 314-701, Chungnam, Korea
E-mail address: dyjeon@kongju.ac.kr

Korea Advanced Institute for Science and Technology, Daejeon 305-701, Korea
E-mail address: sykang@kias.re.kr

Hanyang University, Seoul 133-791, Korea
E-mail address: chhkim@hanyang.ac.kr